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Weak Noise Expansions through Functional Integrals for Colored Noise

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We use path integral methods to obtain expansions for the correlation functions of the non-Markovian stochastic processes generated by stochastic differential equations with colored noise.

KEY WORDS: Colored noise; non-Markovian processes; stochastic differential equations; perturbation expansions.

1. INTRODUCTION

The study of systems in the presence of colored noise has been recently reviewed by Sancho and San Miguel[(1)] and van Kampen.[(2)] In these papers attention has been focused mainly on equations for the probability densities and on the stationary probability of the non-Markovian stochastic process. Here we shall study correlation functions for such processes and give perturbation expansions for them in formal power series in a parameter measuring the strength of the colored noise. We shall use the technique of functional integration to obtain our results through suitable generalizations of the method developed in ref. 3. Path integrals have been used for these processes by Wio *et al.*[(4–6)] in the study of the conditional probability density and of the stationary distribution and similar results have also been obtained by McKane *et al.*[(7,8)] Weak noise expansions for the stationary distribution using WKB-type techniques have been reported in refs. 9 and 10.

In Section 2 we present the formalism for stochastic differential equations of one variable. Explicit expressions are given for the generating

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functional of correlation functions both for the time-dependent regime and for the stationary state. In Section 3 we apply the formalism to an explicit example. In Appendix A we derive once again the basic formula of Section 2 from the two-variable Markov process associated to the original system when the colored noise is the Ornstein–Uhlenbeck process. Although in Section 2 we only treat the one-variable case, the generalization to several variables is possible following the same method.

2. EXPANSIONS OF CORRELATION FUNCTIONS

We consider the equation

$$\dot{x} = A(x) + \sqrt{\eta}\, \sigma(x)\, \xi(t) \tag{1}$$

where $\xi(t)$ is a Gaussian colored noise with mean value zero, $\langle \xi(t) \rangle = 0$, and correlation

$$\langle \xi(t)\, \xi(t') \rangle = \Delta(t, t') = \Delta(t', t) \tag{2}$$

Equation (1) defines a non-Markovian stochastic process when $\Delta(t, t') \neq \delta(t - t')$. In order to construct a perturbation expansion for the correlation functions, we start by giving a functional integral representation for the functional $F[x^{\xi}(\cdot)]$ of the solution $x^{\xi}(t; u_0, t_0)$ of (1) for a given realization of the noise with initial condition $x^{\xi}(t_0) = u_0$. We can write formally

$$F[x^{\xi}(\cdot)] = \int \mathscr{D}Q \prod_{t \in [t_0, T]} \delta(\dot{Q}(t) - A(Q(t)) - \sqrt{\eta}\, \sigma(Q(t))\, \xi(t)) \\ \times F[Q]\, J[Q] \cdot \delta(Q(t_0) - u_0) \tag{3}$$

where $\mathscr{D}Q = \prod_t dQ(t)$ and $J[Q]$ is a Jacobian. We discretize time as $t_j = t_0 + j\varepsilon$, $t_{N+1} = T$, $\varepsilon = (T - t_0)/(N+1)$, and if $f(t)$ is any function of time, we put $f_j = f(t_j)$, $\Delta f_j = f_j - f_{j-1}$. Then the discretized version version of (3) in the prepoint discretization which discretizes $Q(t)$ at the beginning of each interval $[t_{j-1}, t_j]$ is

$$F(x_1^{\xi},..., x_{N+1}^{\xi}) \\ = \int_{Q_0 = u_0} \prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \delta(G_j) \cdot F(Q_1,..., Q_{N+1}) \cdot J(Q_1,..., Q_{N+1}) \tag{4a}$$

$$G_j = \Delta Q_j - \varepsilon A(Q_{j-1}) - \varepsilon \sqrt{\eta}\, \sigma(Q_{j-1}) \xi_j \tag{4b}$$

This expression defines the functional integral (3) as the limit of (4a) when $N \to \infty$ ($\varepsilon \to 0$). The Jacobian in (4a) is $J = |\det B_{kl}|$,

$$B_{kl} = \frac{\partial G_k}{\partial Q_l} = (\delta_{k,l} - \delta_{l,k-1}) - \varepsilon \delta_{l,k-1}(A'(Q_{k-1}) + \sqrt{\eta}\, \sigma'(Q_{k-1})\xi_k) \quad (5)$$

where the prime denotes derivative with respect to the argument. From (5) we see that $B_{kl} = 0$ if $k > l$; then $\det B_{kl} = \prod_l B_{ll} = 1$ and $J = 1$. It should be pointed out that we can discretize in an arbitrary way; for instance, in the discretization $\gamma(\alpha)$ one will have instead of (4b)

$$G_j = \Delta Q_j - \varepsilon A(Q_{j-1} + \alpha \Delta Q_j) - \varepsilon \sqrt{\eta}\, \sigma(Q_{j-1} + \alpha \Delta Q_j)\xi_j$$

and the Jacobian $J \neq 1$, but the final result will be independent of α. This happens due to the following mechanism: the value of the functional integral depends on the discretization and this dependence cancels the effect of the Jacobian, as has been discussed at length in refs. 3, 11, and 12. Using

$$\delta(G_j) = \int \frac{dP_j}{2\pi} \exp(iP_j Q_j)$$

we obtain

$$\begin{aligned} &F(x_1^\xi, \ldots, x_{N+1}^\xi) \\ &\quad = \int_{Q_0 = u_0} \prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{dP_j}{2\pi} \exp i\varepsilon \sum_{j=1}^{N+1} P_j \left(\frac{\Delta Q_j}{\varepsilon} - A(Q_{j-1}) - \sqrt{\eta}\, \sigma(Q_{j-1})\xi_j \right) \\ &\quad \times F(Q_1, \ldots, Q_{N+1}) \end{aligned} \quad (6)$$

The discretized version of the average of a functional $\tilde{G}[\xi(\cdot)]$ over the realizations $\xi(t)$ of the colored noise is

$$\langle \tilde{G}[\xi] \rangle = \int \sum_{i=1}^{N+1} \frac{d\xi_i}{(2\pi)^{1/2}} P(\xi_1, \ldots, \xi_{N+1})\, \tilde{G}(\xi_1, \ldots, \xi_{N+1})) \quad (7a)$$

$$P(\xi_1, \ldots, \xi_{N+1}) = [\det(\Delta^{-1})]^{1/2} \exp\left(-\frac{1}{2} \sum_{j,k} \xi_j \Delta_{jk}^{-1} \xi_k \right) \quad (7b)$$

where Δ^{-1} is the inverse matrix of $\Delta_{jk} = \langle \xi_j \xi_k \rangle = \Delta(t_j, t_k)$, since, using (7), one easily checks that $\langle 1 \rangle = 1$, $\langle \xi_l \rangle = 0$, and $\langle \xi_j \xi_k \rangle = \Delta_{jk}$. If $K_{ij} = K(t_i, t_j)$, where $K(t, t')$ is the inverse kernel of $\Delta(t, t')$ in the sense $\int dt''\, K(t, t'')\, \Delta(t'', t') = \delta(t - t')$, then one has the matrix relation $\varepsilon^2 K = \Delta^{-1}$ and the argument of the exponential in (7b) is $\int dt'\, dt''\, \xi(t')\, K(t', t'')\, \xi(t'')$.

Using (7), we can average (6) over the realization $\xi(t)$, since the integral is a Gaussian. We obtain

$$\langle F(x_1^\xi,\dots, x_{N+1}^\xi)\rangle$$
$$= \int \prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{dP_j}{2\pi} \exp\left[i\varepsilon \sum_{j=1}^{N+1} P_j \left(\frac{\Delta Q_j}{\varepsilon} - A(Q_{j-1})\right)\right.$$
$$\left. - \frac{\eta\varepsilon^2}{2} \sum_{j,k=1}^{N+1} P_j \sigma(Q_{j-1})\, \Delta_{jk} \sigma(Q_{k-1}) P_k\right] \cdot F(Q_1,\dots, Q_{N+1}) \quad (8)$$

We write this as a phase space functional intergral in the $\gamma(0)$ (prepoint) discretization

$$\langle F[x^\xi(\cdot)]\rangle = \int_{\gamma(0)} \mathscr{D}Q\,\mathscr{D}P \exp\left[i \int_{t_0}^{I} dt\, P(\dot{Q} - A(Q))\right.$$
$$\left. - \frac{\eta}{2} \int_{t_0}^{T} dt'\, dt''\, P(t')\, \sigma(Q(t'))\, \Delta(t', t'')\, \sigma(Q(t''))\, P(t'')\right]$$
$$\times F[Q]\, \delta(Q(t_0) - u_0) \quad (9)$$

and this functional integral is defined as the limit when $N \to \infty$ of (8). This is our basic formula and in Appendix A we give another derivation which makes the connection with a two-dimensional Markov process when

$$\Delta(t', t'') = \frac{c}{2\gamma} e^{-\gamma |t' - t''|}$$

From (9) we see that the correlation function $G_m(\tau_1,\dots, \tau_m) = \langle x^\xi(\tau_1) \cdots x^\xi(\tau_m)\rangle$ will be given by

$$G_m(\tau_1,\dots, \tau_m) = \prod_{l=1}^{m} \frac{1}{i} \frac{\delta}{\delta j(t)} \bar{Z}[j, j^*]\Big|_{j=j^*=0} \quad (10)$$

$$\bar{Z}[j, j^*] = \int \mathscr{D}Q\, \mathscr{D}P \exp\left\{ i \int_{t_0}^{T} dt\, [P(\dot{Q} - A(Q)) + j(t)\, Q(t) + j^*(t)\, P(t)]\right.$$
$$\left. - \frac{\eta}{2} \int_{t_0}^{T} dt'\, dt''\, P(t')\, \sigma(Q(t'))\, \Delta(t', t'')\, \sigma(Q(t''))\, P(t'')\right\}$$
$$\times \delta(Q(t_0) - u_0) \quad (11)$$

where T is any time bigger than all the τ_j (we can take $T = +\infty$). From now on we omit [as in (11)] the discretization $\gamma(0)$, since we shall work only in the prepoint discretization.

The perturbation expansion is constructed by making in (11) the change of variables $P(t) \to P(t)/\eta$, $J(t) = \sqrt{\eta}\, j(t)$, $j^*(t) = \sqrt{\eta}\, J^*(t)$, and $Z[J, J^*] = \bar{Z}[J, J^*]$. One has

$$Z[J, J^*] = \int \mathscr{D}Q\, \mathscr{D}^\eta P \exp\left\{\frac{i}{\eta}\int_{t_0}^{T} dt\, [P(\dot{Q} - A(Q)) + \sqrt{\eta}\, JQ + \sqrt{\eta}\, J^* P] \right.$$
$$\left. - \frac{1}{2\eta}\int_{t_0}^{T} dt'\, dt''\, P(t)\, \sigma(Q(t'))\, \Delta(t', t'')\, \sigma(Q(t''))\, P(t'')\right\}$$
$$\times\, \delta(Q(t_0) - u_0) \tag{12}$$

where

$$\mathscr{D}Q\, \mathscr{D}^\eta P = \prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{dP_j}{2\pi\eta}$$

in the discrete. We make now in (12) the change of variables $P(t) = \sqrt{\eta}\, p(t)$, $Q(t) = u(t) + \sqrt{\eta}\, q(t)$, where $u(t; u_0, t_0)$, $u(t_0) = u_0$, is the solution $x = u(t)$ of the deterministic equation [see (1)]

$$\dot{x}(t) = A(x(t)) \tag{13}$$

The terms $O(\eta^{-1})$ in the argument of the exponential in (12) vanish due to the choice of $u(t)$ as well as part of the terms $O(\eta^{-1/2})$. We obtain $(Z[0, 0] = \hat{Z}[0, 0] = 1)$

$$Z[J, J^*] = \exp\left(\frac{i}{\sqrt{\eta}}\int_{t_0}^{T} dt\, J(t)\, u(t)\right) \cdot \hat{Z}[J, J^*] \tag{14a}$$

$$\hat{Z}[J, J^*] = \int \mathscr{D}q\, \mathscr{D}p \exp\left\{ i \int_{t_0}^{T} dt\, [p(\dot{q} - A'(q(t))q) + J(t)\, q(t) + J^*(t)\, p(t)] \right.$$
$$- \frac{1}{2}\int_{t_0}^{T} dt'\, dt''\, p(t)\, \tilde{\Delta}(t, t')\, p(t')$$
$$- i \sum_{n \geqslant 2} \frac{\eta^{(n-1)/2}}{n!} \int_{t_0}^{T} dt\, p(t)\, A^{(n)}(u(t))\, q(t)^n$$
$$- \frac{1}{2} \sum_{n+m \geqslant 1} \frac{\eta^{(n+m)/2}}{n!\, m!} \int_{t_0}^{T} dt\, dt'\, p(t)\, q(t)^n\, \sigma^{(n)}(u(t))$$
$$\left. \times\, \Delta(t, t')\, \sigma^{(m)}(u(t'))\, p(t')\, q(t')^m \right\} \cdot \delta(q(t_0)) \tag{14b}$$

where

$$\mathscr{D}q\,\mathscr{D}p = \prod_{i=1}^{N+1} dq_i \prod_{j=1}^{N+1} \frac{dp_j}{2\pi}$$

in the discrete,

$$\tilde{\Delta}(t, t') = \sigma(u(t))\,\Delta(t, t')\,\sigma(u(t')) = \tilde{\Delta}(t', t)$$

$$A^{(n)}(q) = \frac{d^n A}{dq^n}, \qquad \sigma^{(n)}(q) = \frac{d^n \sigma}{dq^n}$$

Using the generalization of $f(q)e^{Jq} = f(\delta/\delta J)e^{Jq}$, we have

$$\hat{Z}[J, J^*] = K\left[\frac{1}{i}\frac{\delta}{\delta J^*}, \frac{1}{i}\frac{\delta}{\delta J}\right] Z_0[J, J^*] \tag{15a}$$

$$Z_0[J, J^*] = \int \mathscr{D}q\,\mathscr{D}p \exp\left\{i\int_{t_0}^{T} [p(\dot{q} - A'(u(t))q) + J(t)\,q(t) + J^*(t)\,p(t)] \right.$$
$$\left. -\frac{1}{2}\int_{t_0}^{T} dt'\,dt''\,p(t)\,\tilde{\Delta}(t', t'')\,p(t'')\right\} \cdot \delta(q(t_0)) \tag{15b}$$

$$K[p, q] = \exp\left[-i\sum_{n\geqslant 2} \frac{\eta^{(n-1)/2}}{n!} \int_{t_0}^{T} dt\,A^{(n)}(u(t))\,p(t)\,q(t)^n \right.$$
$$-\frac{1}{2}\sum_{n+m\geqslant 1} \frac{\eta^{(n+m)/2}}{n!\,m!} \int_{t_0}^{T} dt\,dt'\,p(t)\,q(t)^n\,\sigma^{(n)}(u(t))$$
$$\left. \times \Delta(t, t')\,\sigma^{(m)}(u(t'))\,p(t')\,q(t')^m\right] \tag{15c}$$

The correlation functions are now [see (10)]

$$G_m(\tau_1,\ldots,\tau_m) = \prod_{l=1}^{m} \frac{\sqrt{\eta}}{i} \frac{\delta}{\delta J(\tau_l)} Z[J, J^*]\Big|_{J=J^*=0} \tag{16}$$

and since $Z_0[J, J^*]$ is independent of η [all the η dependence is explicitly shown in (14) and (15)], we can see from (14) and (15) that they can be calculated as formal power series in η. We see in (15b) that Z_0 is given by a Gaussian functional integral which can be calculated (see Appendix B, $Z_0[0, 0] = 1$)

$$Z_0[J, J^*] = \exp\left[-i\int_{t_0}^{T} dt\,dt'\,J(t)\,S(t, t')\,J^*(t') \right.$$
$$\left. -\tfrac{1}{2}\int_{t_0}^{T} dt\,dt'\,J(t)\,D(t, t')\,J(t')\right] \tag{17a}$$

$$S(t, t') = \theta(t-t') \exp \int_{t'}^{t} d\tau\, A'(u(\tau)) \tag{17b}$$

$$D(t, t') = \int_{t_0}^{T} dt_1\, dt_2\, S(t, t_1)\, \tilde{\Delta}(t_1, t_2)\, S(t', t_2) = D(t', t) \tag{17c}$$

Here $\theta(t)$ is the step function, $\theta(t)=1$, $t>0$, and $\theta(t)=0$, $t<0$. In the calculation of $\hat{Z}[J, J^*]$ by (15a), one finds terms of the form

$$\frac{\delta}{\delta J^*(t)} \frac{\delta}{\delta J(t)} Z_0[J, J^*] = -i\theta(0) \tag{18}$$

which are undefined. But the prepoint discretization $\gamma(0)$ tells us that they must be interpreted as

$$\lim_{\varepsilon \to +0} \frac{\delta}{\delta J^*(t+\varepsilon)} \frac{\delta}{\delta J(t)} Z_0[J, J^*] = -i\theta(-\varepsilon) = 0 \tag{19}$$

since $p(t)\, q(t)$ is discretized as $p_j q_{j-1}$. As an example, we calculate the mean value $G_1(\tau) = \langle x(\tau) \rangle$ using (14)–(16) in the case of $\sigma(q)=1$. We obtain

$$G_1(\tau) = \frac{\sqrt{\eta}}{i} \frac{\delta Z}{\delta J(\tau)}\bigg|_{J=J^*=0}$$

$$= u(\tau) + \frac{\eta}{2} \int_{t_0}^{\tau} dt\, A^{(2)}(u(t))\, S(\tau, t)\, D(t, t) + O(\eta^2) \tag{20}$$

The formulas obtained up to now give the fluctuations around the deterministic trajectory $x = u(t; u_0, t_0)$. We consider now the stationary state which is obtained taking the limit $t_0 \to -\infty$ in (14)–(17). We have then to examine the attractors of the dynamical system $\dot{x} = A(x(t))$ [see (13)]. Let μ be an attractor, i.e., $A(\mu)=0$, $A'(\mu)<0$, and $B(\mu)$ its basin of the attraction, i.e., if u_0 is in $B(\mu)$, one has that $u(t; u_0, t_0) = \tilde{u}(\tau;\ u_0)$, $\tau = t - t_0$, tends to μ when $\tau \to \infty$. The generating functional $Z_0^{\text{stat}}[J, J^*]$ corresponding to this attractor is obtained from (17),

$$Z_0^{\text{stat}}[J, J^*] = \exp\left[-i \int_{-\infty}^{\infty} dt\, dt'\, J(t)\, S^{\text{st}}(t, t')\, J^*(t') \right.$$

$$\left. - \tfrac{1}{2} \int_{-\infty}^{\infty} dt\, dt'\, J(t)\, D^{\text{st}}(t, t')\, J(t') \right] \tag{21a}$$

$$S^{\text{st}}(t, t') = \theta(t-t') \exp[A'(\mu)(t-t')] \tag{21b}$$

$$D^{\text{st}}(t, t') = \sigma(\mu)^2 \int_{-\infty}^{\infty} dt_1\, dt_2\, S^{\text{st}}(t, t_1)\, \Delta(t_1, t_2)\, S^{\text{st}}(t', t_2) \tag{21c}$$

We shall specify now the colored noise ξ as an Ornstein–Uhlenbeck process with correlation ($c > 0$, $\gamma > 0$)

$$\langle \xi(t)\, \xi(t') \rangle = \Delta(t-t') = \frac{c}{2\gamma} e^{-\gamma |t-t'|} \tag{22}$$

In this case D^{st} has the value $[b = -A'(\mu) < 0]$

$$D^{\mathrm{st}}(t, t') = \frac{c\sigma(\mu)^2}{2\gamma} \left[\frac{1}{b(b+\gamma)} e^{-b|t-t'|} + \frac{1}{b^2-\gamma^2} (e^{-\gamma|t-t'|} - e^{-b|t-t'|}) \right] \tag{23}$$

The white noise limit is here $c = \gamma^2$, $\gamma \to \infty$, which gives $\Delta(t) \to \delta(t)$ and (23) tends to the usual result $[\sigma(\mu)^2/2b] \exp(-b|t-t'|)$. There is no singularity in (23) at $b^2 = \gamma^2$. We remark that the expansion for the stationary correlation functions is local in the sense that one will obtain a system of correlation functions for each attractor of the deterministic dynamical system $\dot{x} = A(x)$. If there is only attractor, the result is global. In the case of coexistence of attractors the fluctuations that we determine around one of them have a meaning if the escape time from that local attractor is much bigger than the times in which we are interested. Since the dominant behavior of the escape time is of the form $\exp(r/\eta)$ with r a positive constant, the expansion is asymptotically valid for $\eta \to 0$.

An alternative way of doing the calculations is to define a new stochastic process $q(t)$ by making in (1) the change of variables $x(t) = u(t) + \sqrt{\eta}\, q(t)$, which gives $[\sigma^{(0)}(u(t)) \equiv \sigma(u(t))]$

$$\begin{aligned} \dot{q}(t) = \sum_{n \geq 1} \frac{\eta^{(n-1)/2}}{n!} A^{(n)}(u(t))\, q(t)^n \\ + \sum_{n \geq 0} \frac{\sigma^{(n)}(u(t))}{n!} [\sqrt{\eta}\, q(t)]^n \, \xi(t) \end{aligned} \tag{24}$$

The generating functional of correlation functions $\tilde{G}_m(\tau_1, ..., \tau_m) = \langle q(\tau_1) \cdots q(\tau_m) \rangle$ of the process $q(t)$ is

$$\begin{aligned} \tilde{Z}[J, J^*] = \int \mathscr{D}q\, \mathscr{D}p \exp \bigg\{ i \int_{t_0}^{T} dt\, [p(\dot{q} - A^{(1)}(u(t))q) \\ + J(t)\, q(t) + J^*(t)\, p(t)] \\ - \tfrac{1}{2} \int_{t_0}^{T} dt\, dt'\, p(t)\, \sigma(u(t))\, \Delta(t, t')\, \sigma(u(t'))\, p(t') \bigg\} \\ \times K[p, q] \cdot \delta(q(t_0)) \end{aligned} \tag{25}$$

where $K[p, q]$ is the functional given by (15c). One has

$$\tilde{G}_m(\tau_1,\dots,\tau_m)=\prod_{l=1}^{m}\frac{1}{i}\frac{\delta}{\delta J(\tau_l)}\tilde{Z}[J, J^*]\Big|_{J=J^*=0} \tag{26}$$

Comparing (15a) with (25), we see that $\tilde{Z}[J, J^*]=\hat{Z}[J, J^*]$.

3. APPLICATIONS OF THE FORMALISM

Correlation functions and cumulants of the original process $x(t)$ are related to the corresponding quantities of $q(t)$ through the transformation $x(t)=u(t)+\sqrt{\eta}\,q(t)$ and consequently we can work with $q(t)$. Using (15) and (26), we obtain the correlation functions of $q(t)$:

$$\begin{aligned}\tilde{G}_m(\tau_1,\dots,\tau_m)&=\langle q(\tau_1)\cdots q(\tau_m)\rangle\\ &=\prod_{l=1}^{m}\frac{1}{i}\frac{\delta}{\delta J(\tau_l)}K\left[\frac{1}{i}\frac{\delta}{\delta J^*},\frac{\delta}{\delta J}\right]Z_0[J, J^*]\Big|_{J=J^*=0}\end{aligned} \tag{27}$$

Using the generalization of the formula

$$F\left(\frac{1}{a}\frac{\partial}{\partial J}\right)Z(J)\Big|_{J=0}=Z\left(\frac{1}{a}\frac{\partial}{\partial q}\right)F(q)\Big|_{q=0}$$

valid for any functions $F(\cdot)$ and $Z(\cdot)$, we obtain from formula (27) the basic equation

$$\tilde{G}_m(\tau_1,\dots,\tau_m)=Z_0\left[\frac{1}{i}\frac{\delta}{\delta q},\frac{1}{1}\frac{\delta}{\delta p}\right]q(\tau_1)\cdots q(\tau_m)\,K[p, q]\Big|_{p=q=0} \tag{28}$$

Here Z_0 is independent of η and $K[p, q]$ has an expansion in powers of $\sqrt{\eta}$, but inspection of (28) shows that for m even one has only powers $\sqrt{\eta}^{2n}=\eta^n$ and for m odd only powers $\sqrt{\eta}^{2n+1}=\sqrt{\eta}\eta^n$. Each term in (28) is a multiple integral with an integrand which is the product of given functions of time with quantities of the form

$$Z_0\left[\frac{1}{i}\frac{\delta}{\delta q},\frac{1}{i}\frac{\delta}{\delta p}\right](p(\tau_1)\,p(\tau_2)\cdots q(\tau_1')\,q(\tau_2')\cdots)\Big|_{p=q=0}=\{p(\tau_1)\cdots q(\tau_1')\cdots\} \tag{29}$$

where the notation $\{\cdots\}$ is defined by (29). Putting $z_1(\tau)=p(\tau)$, $z_2(\tau)=q(\tau)$, we define the contractions $\overline{z_\mu(t)\,z_\nu}(t')\equiv\{z_\mu(t)\,z_\nu(t')\}$, which have the values

$$\overline{z_\mu(t)\, z}_\nu(t') = Z_0\left[\frac{1}{i}\frac{\delta}{\delta q}, \frac{1}{i}\frac{\delta}{\delta p}\right] z_\mu(t)\, z_\nu(t')\Bigg|_{p=q=0} = \Delta_{\mu\nu}(t, t') \tag{30a}$$

$$\begin{aligned} \Delta_{11}(t, t') &= 0, & \Delta_{12}(t, t') &= iS(t', t) \\ \Delta_{21}(t, t') &= iS(t, t'), & \Delta_{22}(t, t') &= D(t, t') \end{aligned} \tag{30b}$$

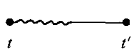
Due to the form of $Z_0[J, J^*]$, which is a exponential of a quadratic form in (J, J^*), one can easily see that the quantities $\{p(\tau_1)\cdots q(\tau_1')\cdots\}$ in (29) have the values [we put $z_j = z(\tau_j)$] $\{z_1 z_2 \cdots z_{2n+1}\} = 0$ and

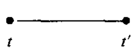
$$\{z_1 z_2 \cdots z_{2n}\} = \overline{z_1 z_2}\, \overline{z_3 z_4} \cdots \overline{z_{2n-1} z_{2n}} + \text{(all possible pairs)} \tag{31}$$

which is a sum over all possible contractions of which there are $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ (this formula is called the Wick theorem; see, for example, ref. 14). When $n=2$ one has the $3!! = 3$ terms

$$\{z_1 z_2 z_3 z_4\} = \overline{z_1 z_2}\, \overline{z_3 z_4} + \overline{z_1 z_3}\, \overline{z_2 z_4} + \overline{z_1 z_4}\, \overline{z_2 z_3} \tag{32}$$

Each term in (31) has a graphical representation. We put

$\overline{z_1(t)\, z}_2(t') = \overline{p(t)\, q}(t') =$  $= iS(t, t')$

$\overline{z_2(t)\, z}_2(t') = \overline{q(t)\, q}(t') =$  $= D(t, t')$

For example, in the sum over contractions of $\{q(t')\, q(t'')\, p(\tau)\, q(\tau)^3\}$ we have the term

$$\overline{q(t')\, q(t'')\, p}(\tau)\, q(\tau)\, \overline{q(r)\, q}(\tau) = iS(t', \tau)\, D(t'', \tau)\, D(\tau, \tau) \tag{33}$$

which has the graph in Fig. 1.

The functional $\tilde{Z}[J, 0]$ generates the correlations

$$\tilde{G}_m(\tau_1, \ldots, \tau_m) = \prod_{l=1}^{m} \frac{1}{i}\frac{\delta}{\delta J(\tau_l)}\, \tilde{Z}[J, 0]\Bigg|_{J=0}$$

and $W[J] = \ln \tilde{Z}[J, 0]$ the cumulants $C_m(\tau_1, \ldots, \tau_m) \equiv \langle\!\langle q(\tau_1)\cdots q(\tau_m)\rangle\!\rangle$ by

$$C_m(\tau_1, \ldots, \tau_m) = \prod_{l=1}^{m} \frac{1}{i}\frac{\delta}{\delta J(\tau_l)}\, W[J]\Bigg|_{J=0} \tag{34}$$

It is simple to see that $C_m(\tau_1, \ldots, \tau_m)$ can be obtained from (27) keeping only the connected graphs (these are the graphs which are not formed of two or more separated parts) in the right-hand side. Using now these techniques,

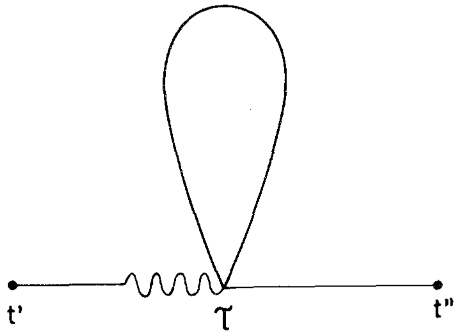


Fig. 1. Graph contributing to the correlation function.

we shall give explicit expressions for the two-point cumulant $C_2(t, t') = \langle\langle q(t')\, q(t'') \rangle\rangle$ corresponding to (1) when $\sigma(x) = 1$. In this case we have

$$K[p, q] = \exp \sqrt{\eta} \int_{t_0}^{T} dt\, H_I(t)$$

with

$$H_I(t) = \sum_{n \geq 0} \eta^{n/2} H_I^{(n)}(t), \qquad H_I^{(n)} = \frac{1}{i(n+2)!} A^{(n+2)}(u(t))\, pq^{n+2} \tag{35}$$

Putting $C_2(t', t'') = R_0(t', t'') + \eta R_1(t', t'') + O(\eta^2)$, one has $R_0(t', t'') = D(t', t'')$ and

$$R_1(t', t'') = R_1^{(1)}(t', t'') + R_1^{(2)}(t', t'') \tag{36a}$$

$$R_1^{(1)}(t', t'') = \int_{t_0}^{T} d\tau\, \{q(t')\, q(t'')\, H_I^{(1)}(\tau)\} \tag{36b}$$

$$R_1^{(2)}(t', t'') = \tfrac{1}{2} \int_{t_0}^{T} d\tau_1\, d\tau_2\, \{q(t')\, q(t'')\, H_I^{(0)}(\tau_1)\, H_I^{(0)}(\tau_2)\} \tag{36c}$$

Using the Wick theorem [formula (31)], we obtain

$$R_1^{(1)}(t', t'') = \tfrac{1}{2} \int_{t_0}^{t'} d\tau\, A^{(3)}(u(\tau))\, S(t', \tau)\, D(t'', \tau)\, D(\tau, \tau) + (t' \leftrightarrow t'') \tag{37}$$

which corresponds to the graph in Fig. 1 and where $(t' \leftrightarrow t'')$ means we have to add the previous term interchanging t' and t'' and

$$R_1^{(2)}(t',t'')=\tfrac{1}{2}\int_{t_0}^{t'} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\, A^{(2)}(u(\tau_1))\, A^{(2)}(u(\tau_2))$$
$$\times S(t',\tau_1)\, S(\tau_1,\tau_2)\, D(\tau_1,t'')\, D(\tau_2,\tau_2)$$
$$+\int_{t_0}^{t'} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\, A^{(2)}(u(\tau_1))\, A^{(2)}(u(\tau_2))$$
$$\times S(t',\tau_1)\, S(\tau_1,\tau_2)\, D(\tau_1,\tau_2)\, D(t'',\tau_2)$$
$$+\tfrac{1}{4}\int_{t_0}^{t'} d\tau_1 \int_{t_0}^{t''} d\tau_2\, A^{(2)}(u(\tau_1))\, A^{(2)}(u(\tau_2))$$
$$\times S(t',\tau_1)\, S(t'',\tau_2)\, D(\tau_1,\tau_2)^2 + (t' \leftrightarrow t'') \tag{38}$$

We specialize now to $A(x)=\lambda x - x^3$, where $A^{(1)}=\lambda-3x^2$, $A^{(2)}=-6x$, $A^{(3)}=-6$, $A^{(n)}=0$, $n\geqslant 4$, and

$$u(t)=\pm\left\{\frac{\lambda u_0^2}{u_0^2+(\lambda-u_0^2)\exp[-2\lambda(t-t_0)]}\right\}^{1/2} \tag{39}$$

where the plus sign corresponds to $u_0>0$ and the minus sign to $u_0<0$. For $\lambda<0$ and for any initial condition $u(\tau=t-t_0)\to 0$, $\tau\to\infty$. For $\lambda>0$ one has $u(\tau)\to\sqrt{\lambda}$ for $u_0>0$ and $u(\tau)\to -\sqrt{\lambda}$ for $u_0<0$ (we have bistability). Replacing $u(t)$ in (36)–(38), we obtain $C_2(t',t'')$. For the stationary state we have to distinguish according to the sign of λ.

For $\lambda<0$ one has $b=-A^{(1)}(0)=-\lambda$ [see (21)–(23)], $A^{(2)}(0)=0$, $A^{(3)}(0)=-6$, and

$$S^{\rm st}(t,t')=\theta(t-t')e^{\lambda(t-t')} \tag{40a}$$

$$D^{\rm st}(t,t')=\frac{c}{2\gamma}\left[\frac{1}{\lambda(\gamma-\lambda)}e^{\lambda|t-t'|}+\frac{1}{\lambda^2-\gamma^2}(e^{-\gamma|t-t'|}-e^{\lambda|t-t'|})\right] \tag{40b}$$

Replacing in (37)–(38), one has $R_1^{(2)}=0$ and

$$R_1^{(1)}(t',t'')$$
$$=-3\left(\frac{c}{2\gamma(\lambda^2-\gamma^2)}\right)^2$$
$$\times\left[\frac{2}{\gamma-\lambda}e^{-\gamma|t'-t''|}+\frac{3\lambda^2\gamma-\gamma^3}{\lambda^3(\gamma-\lambda)}e^{\lambda|t'-t''|}+\frac{\lambda+\gamma}{\lambda^2}\gamma|t'-t''|e^{\lambda|t'-t''|}\right] \tag{41}$$

For $\lambda>0$ we take $u_0>0$; then $u(\tau)\to\sqrt{\lambda}$, $A^{(1)}(\sqrt{\lambda})=-2\lambda$, $A^{(2)}(\sqrt{\lambda})=-6\sqrt{\lambda}$, $A^{(3)}(\sqrt{\lambda})=-6$, and

$$S^{st}(t,t') = \theta(t-t')e^{-2\lambda(t-t')} \tag{42a}$$

$$D^{st}(t,t') = \frac{c}{2\gamma}\left[\frac{1}{2\lambda(2\lambda+\gamma)}e^{-2\lambda|t-t'|} + \frac{1}{4\lambda^2-\gamma^2}\left(e^{-\gamma|t-t'|} - e^{-2\lambda|t-t'|}\right)\right] \tag{42b}$$

Replacing in (37)–(38), we obtain now

$$\begin{aligned} R_1^{(1)}(t',t'') &= -3\left(\frac{c}{2\gamma(4\lambda^2-\gamma^2)}\right)^2 \\ &\times\left[\frac{2}{2\lambda+\gamma}e^{-\gamma|t'-t''|} - \frac{12\lambda^2\gamma-\gamma^3}{8\lambda^3(2\lambda+\gamma)}e^{-2\lambda|t'-t''|} - \frac{2\lambda-\gamma}{4\lambda^2}\gamma|t'-t''|e^{-2\lambda|t'-t''|}\right] \end{aligned} \tag{43a}$$

$$\begin{aligned} R_1^{(2)}(t',t'') &= 9\left(\frac{c}{2\gamma(4\lambda^2-\gamma^2)}\right)^2\left[\frac{24\lambda^3+24\lambda^2\gamma+7\lambda\gamma^2+\gamma^3}{\lambda(\lambda+\gamma)(2\lambda+\gamma)(4\lambda+\gamma)}e^{-\gamma|t'-t''|}\right. \\ &+\frac{128\lambda^6+48\lambda^5\gamma-276\lambda^4\gamma^2-234\lambda^3\gamma^3-40\lambda^2\gamma^4+5\lambda\gamma^5+\gamma^6}{8\lambda^3(\lambda+\gamma)(2\lambda+\gamma)(4\lambda+\gamma)\gamma}e^{-2\lambda|t'-t''|} \\ &-\frac{8\lambda^2+2\lambda\gamma-3\gamma^2}{4\lambda^2}|t'-t''|e^{-2\lambda|t'-t''|} + \frac{\gamma}{\lambda(4\lambda-\gamma)}\left(e^{-\gamma|t'-t''|}-e^{-4\lambda|t'-t''|}\right) \\ &-\frac{1}{4(\lambda-\gamma)}\left(e^{-2\lambda|t'-t''|}-e^{-2\gamma|t'-t''|}\right)+\frac{1}{4(\lambda+\gamma)}e^{-2\gamma|t'-t''|} \\ &\left.-\frac{8\lambda^2-2\lambda\gamma-4\gamma^2}{(\lambda+\gamma)(4\lambda+\gamma)\gamma}e^{-(2\lambda+\gamma)|t'-t''|}+\frac{8\lambda^2+4\lambda\gamma+\gamma^2}{8\lambda^3(4\lambda+\gamma)}\gamma e^{-4\lambda|t'-t''|}\right] \end{aligned} \tag{43b}$$

For $u_0 < 0$ one has $u(\tau) \to -\sqrt{\lambda}$ and the final result is again (42)–(43). We remark that the above expressions are not singular at $\gamma^2 = \lambda^2$, $\gamma^2 = 4\lambda^2$, or $\gamma^2 = 4\lambda$ [see also (23)].

APPENDIX A

We consider here the colored noise $\xi(t)$ with $\langle\xi(t)\,\xi(t')\rangle$ give by (22), which is just the stationary correlation function of the Ornstein–Uhlenbeck process defined by the equation $\dot{\xi}(t) = -\gamma\xi + \sqrt{c}\,f(t)$, where $f(t)$ is a δ-correlated white noise of zero mean, i.e., $f(t)$ is a Gaussian process with $\langle f(t)\rangle = 0$, $\langle f(t)\,f(t')\rangle = \delta(t-t')$. We put $q_1(t) = q(t)$, $q_2(t) = \xi(t)$, and consider instead of (1) the coupled stochastic differential equations

$$\dot{q}_1 = A(q_1) + \sqrt{\eta}\,\sigma(q_1)q_2 \tag{A1}$$

$$\dot{q}_2 = -\gamma q_2 + \sqrt{c}\,f(t) \tag{A2}$$

The generating function $Z_1[j, j^*]$ of correlation and response functions for the process $q_1(t)$ is given by[3,13]

$$Z_1[j, j^*] = \int_{\gamma(0)} \mathscr{D}\mathbf{q}\, \mathscr{D}\mathbf{p} \exp i \int_{t_0}^{T} dt \left[p_1(\dot{q}_1 - A(q_1) - \sqrt{\eta}\, \sigma(q_1) q_2) \right.$$

$$\left. + p_2(\dot{q}_2 + \gamma q_2) + \frac{ic}{2} p_2^2 + j q_1 + j^* p_1 \right]$$

$$\times \delta(q_1(t_0) - u_0)\, \delta(q_2(t_0) - \beta) \tag{A3}$$

$$\mathscr{D}\mathbf{q}\, \mathscr{D}\mathbf{p} = \mathscr{D}q_1\, \mathscr{D}q_2\, \mathscr{D}p_1\, \mathscr{D}p_2 = \prod_{i=1}^{N+1} dq_{1,i}\, dq_{2,i} \prod_{j=1}^{N+1} \frac{dp_{1,j}}{2\pi} \frac{dp_{2,j}}{2\pi}$$

in the discrete and we have taken initial conditions $q_1(t_0) = u_0$ and $q_2(t_0) = \beta$. We can write (A3) as [we omit $\gamma(0)$]

$$Z_1[j, j^*] = \int \mathscr{D}q_1\, \mathscr{D}q_1 \exp i \int_{t_0}^{T} dt\, [p_1(\dot{q}_1 - A(q_1)) + j q_1 + j^* p_1]$$

$$\times \delta(q_1(t_0) - u_0) \cdot Z_2[-\sqrt{\eta}\, \sigma p_1, 0] \tag{A4}$$

$$Z_2[J, J^*] = \int \mathscr{D}q_2\, \mathscr{D}p_2 \exp i \int_{t_0}^{T} dt \left[p_2(\dot{q}_2 + \gamma q_2) + \frac{ic}{2} p_2^2 + J q_2 + J^* p_2 \right]$$

$$\times \delta(q_2(t_0) - \beta) \tag{A5}$$

But Z_2 is the generating functional for the Ornstein–Uhlenbeck process and can be calculated since it is a Gaussian functional integral. Using the boundary conditions

$$\left. \frac{\delta Z_2}{\delta J(t)} \right|_{t = t_0} = \beta Z_2, \qquad \left. \frac{\delta Z_1}{\delta J^*(t)} \right|_{t = T} = 0, \qquad Z_2[0, 0] = 1$$

one obtains the result[3,13]

$$Z_2[J, J^*] = \exp \left[i\beta \int_{t_0}^{T} dt\, e^{\gamma(t_0 - t)} J(t) \right.$$

$$- i \int dt'\, dt''\, J(t)\, \bar{S}(t' - t'')\, J(t'')$$

$$\left. - \frac{1}{2} \int_{t_0}^{T} dt'\, dt''\, J(t')\, \bar{\Delta}(t', t'')\, J(t'') \right] \tag{A6}$$

$$\bar{S}(t) = \theta(t) e^{-\gamma t} \tag{A7}$$

$$\bar{\Delta}(t, t') = \frac{c}{2\gamma} (e^{-\gamma |t - t'|} - e^{-\gamma(t + r') + 2\gamma t_0}) \tag{A8}$$

The next step is to do the average of $Z_1[j, j^*]$ in (A4) over the initial condition $q_2(t_0)=\beta$ with the stationary probability density of this process

$$p_{st}(\beta)=\left(\frac{\gamma}{\pi c}\right)^{1/2}\exp\left(-\frac{\gamma}{c}\beta^2\right)$$

since we are assuming that the colored noise is in the stationary state. This means that the time scale γ^{-1} defined by the colored noise is smaller than the other relevant scales [here the relaxation time $|A'(\mu)|^{-1}$ of q_1]. We have

$$\int d\beta\, Z_2[J, J^*=0]\, p_{st}(\beta)$$

$$=\exp\left(-\frac{1}{2}\int_{t_0}^{T} dt'\, dt''\, J(t')\frac{c}{2\gamma}e^{-\gamma|t'-t''|}J(t'')\right) \tag{A9}$$

Using this result for $J(t)=-\sqrt{\eta}\,\sigma(u(t))\,p_1(t)$ in $\int d\beta\, Z_1[j, j^*]\, p_{st}(\beta)=\bar{Z}[j, j^*]$, we obtain

$$\bar{Z}[j, j^*]=\int \mathscr{D}q_1\,\mathscr{D}p_1 \exp\left[i\int_{t_0}^{T} dt\,[p_1(\dot{q}_1-A(q_1))+jq_1+j^*p_1]\right.$$

$$\left.-\frac{\eta}{2}\int_{t_0}^{T} dt'\, dt''\, p_1(t')\,\sigma(u(t'))\frac{c}{2\gamma}e^{-\gamma|t'-t''|}\sigma(u(t''))\,p_1(t'')\right]$$

$$\times\,\delta(q_1(t_0)-u_0) \tag{A10}$$

which was the starting point (11) of our method.

APPENDIX B

We shall calculate here the Gaussian functional integral (15b). Using the integration by parts lemma

$$\int \mathscr{D}q\,\mathscr{D}p\,\frac{\delta}{\delta q(t)}B[p, q]=\int \mathscr{D}q\,\mathscr{D}p\,\frac{\delta}{\delta p(t)}B[p, q]=0$$

for any functional $B[p, q]$ we obtain from (15b) for $B[p, q]$ equal to the integrand the equations

$$\int \mathscr{D}q\,\mathscr{D}p\,\{-\dot{p}(t)-A'(u(t))+J(t)\}\,B[p, q]=0 \tag{B1}$$

$$\int \mathscr{D}q\,\mathscr{D}p\left\{\dot{q}(t)-A'(u(t))+i\int_{t_0}^{T} dt'\,\tilde{\Delta}(t, t')\,p(t')+J^*(t)\right\}B[p, q]=0 \tag{B2}$$

These equations give two differential equations

$$\left\{\frac{\partial}{\partial t} - A'(u(t))\right\} \frac{1}{i} \frac{\delta Z_0}{\delta J(t)} = -\int_{t_0}^{T} dt' \, \tilde{\Delta}(t, t') \frac{\delta Z_0}{\delta J^*(t)} - J^*(t) Z_0 \qquad \text{(B3)}$$

$$\left\{\frac{\partial}{\partial t} + A'(u(t))\right\} \frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)} = J(t) Z_0 \qquad \text{(B4)}$$

which have to be solved with the boundary conditions

$$\left.\frac{1}{i} \frac{\delta Z_0}{\delta J(t)}\right|_{t=t_0} = \left.\frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)}\right|_{t=T} = 0, \qquad Z_0[0, 0] = 1 \qquad \text{(B5)}$$

From (B4) and (B3) we obtain

$$\frac{1}{i} \frac{\delta Z_0}{\delta J^*(t)} = -\int_{t_0}^{T} dt' \, J(t') \, S(t', t) \cdot Z_0 \qquad \text{(B6)}$$

$$\frac{1}{i} \frac{\delta Z_0}{\delta J(t)} = \left\{ -\int_{t_0}^{T} dt' \, S(t, t') \, J^*(t') + i \int_{t_0}^{T} dt' \, J(t') \, D(t', t) \right\} \cdot Z_0 \qquad \text{(B7)}$$

with $S(t, t')$ and $D(t', t)$ given by (17). From (B6) and (B7) one immediately obtains (17a).

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